Theory of Laplace and Fourier Transform With Its Applications

J.R.Sontakke
Department Of Mathematics, Datta Meghe Institute Of Engineering Technology And Research (Wardha) M.S.

Abstract – The concept of Laplace Transformation and Fourier Transformation play a vital role in various areas of science and engineering. In solving problems relating to these fields, one usually encounters problems on time invariants, differential equations, time and frequency domains for non-periodic wave forms. This paper will discuss the fundamentals of Laplace transform and Fourier transform and basic applications of these fundamentals to electric circuits and signal design and solution to related problems.

Index Terms – Laplace Transform & Fourier Transform.

1. INTRODUCTION

Laplace transform is an integral transform method which is particularly useful in solving linear ordinary differential equation. It finds very wide applications in various areas of physics , electrical engineering, control engineering, optics, mathematics , signal processing. The laplace transform can be interpreted as a transformation from the time to the frequence domain where inputs and outputs are function of time f(t) to be laplace transformable, it must satisfy the following Dirichlet conditions :

1) *f*(*t*) must be piecewise continuous which means that it must be single valued but can have a finite number of finite isolated discontinuities for $t > 0$.

2) *f*(*t*) must be exponential order which means that *f*(*t*) must remain less than $se^{-a_0 t}$ as t approaches ∞ where *S* is a positive constant and a_0 is a real positive number.

If there is any function *f*(*t*) that satisfies the Dirichlet Conditions then,

$F(s) = \int_0^\infty f(t)e^{-st}dt$ written as *L*(*f*(*t*)) is called the laplace transformation of f(t). Here, s can be either a real variable or a complex quantity.

The integral $\int_0^\infty f(t)e^{-st}dt$ converges if $\int |f(t)e^{-st}|dt < \infty$, $s = \sigma + jw$ [1][2]

Fourier transform is used for decomposing signals into its constituent frequencies and its oscillatory functions. It represent signal in frequency domain and transforms one complex value function of a real variable into another.

Fourier transform entails representation of a non-periodic function not as a sum but as an integral over a continuous range of frequencies.[8][5]. This is done by converting infinite Fourier series in terms of series and cosines into a double infinite series involving complex exponentials. On other hands, Fourier transform basically involve frequency domain representation of non-periodic function, in which such representation is valid over the entire time domain and accomplished by means of Fourier integral[7][5].

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\left[\int_{-\infty}^{\infty} f(\tau)e^{-i\omega t}d\tau\right]d\omega$$

It is reduces to

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)e^{i\omega t}d\omega$$

2. PROPERTIES OF LAPLACE TRANSFORM

Some of the very important properties of Laplace transforms which will be used in its applications to be discussed later on are described as follows: [3]

2.1. Linearity

If Laplace Transform of function f(t),g(t) and h(t) be $\overline{f(s)}, \overline{g(s)} and\ \bar{h}(S)$ respectively , then

L$\{af(t) + bg(t) + h(t)\} = a\bar{f}(s) + b\bar{g}\,(s) + c\bar{h}\,(s)$ Where a, b, c are constant.

2.2. Differentiation

If L$\{f(t)\} = \bar{f}(s)$, then $L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0)\ - - f^{n-1}(0)$

Where n denote n^{th} derivative

2.3 Integral

If L$\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\} = \frac{\bar{f}(s)}{s}$

2.4 Mutipication by t

If L$\{f(t)\} = \bar{f}(s)$, then

(i) $L\{tf(t)\} = -\frac{d}{ds}\bar{f}(s)$

(ii) $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$

2.5 Division by t

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{\boldsymbol{f}}(\boldsymbol{s})ds$

2.6 Time delay

The substitution of $t - \lambda$ for the variable t in the transform $Lf(t)$ corresponds to the multiplication of the function $F(s)$ by $e^{-\lambda s}$ that is

$$L\big(f(t-\lambda)\big) = e^{-\lambda s}F(s)\ .\ [2]$$

2.7 Periodic function

A function f(t) is said to be periodic if and only if there exists a positive real number T such that for all values of t,

$$f(t) = f(t+T) = f(t+2t) = \ --= f(t+nT)$$

The positive real number T is called the period of the function f(t) and it is the least value of variable t by which it may be increased or decreased without affecting the value of the function f(t).

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt.$$

2.8 Unit step function

The unit step function can be said to be the force of magnitude 1 which is applied to a system at time $t \geq 0$ and it is defined as

$$U(t) = \begin{cases} 0\ ,\ t < 0 \\ 1\ , t \geq 0 \end{cases}$$

$$L[U(t)] = \frac{1}{s}\ ,\ s > 0$$

2.9 Transfer function

Mathematically it is defined as the ratio of Laplace transform of output (response) of the system to the Laplace transform of input (driving function) , under the assumption that all initial conditions are zero.[4]

If T(s) is the transfer function of the system then ,

$$T(s) = \frac{Laplace\ transform\ of\ output}{Laplace\ transform\ of\ input} = \frac{C(s)}{R(s)}$$

3. APPLICATIONS OF LAPLACE TRANSFORM

In engineering and science, the Laplace transform is used for solving problems of time invariant systems such as electrical circuits, harmonics, oscillations, mechanical system, control theory , optical devices ,signal analysis and design , system analysis, and solving differential equations. The Laplace in its analysis transforms the time domain in which outputs and inputs are function of time to the frequency domain (the inputs and output function of complex angular frequency in radians per unit time). [5]

3.1 Application in electric circuit theory

The Laplace transform can be applied to solve the switching transient phenomenon in the series or parallel RL,RC or RLC circuits [6]. A simple example of showing this application follows next.

Let us consider a series RLC circuit as shown in Fig 1. to which a d.c. voltage *Vo* is suddenly applied.

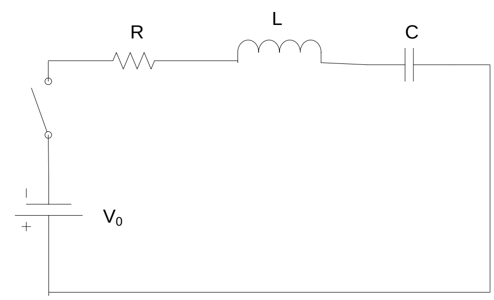

Fig.1 : Series RLC circuit

Now, applying Kirchhoff"s Voltage Law (KVL) to the circuit, we have,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\ dt = V_0\ (3)$$

Differentiating both sides,

$$L\frac{d^2i}{dt^2} + \frac{1}{C}i + R\frac{di}{dt} = 0$$

$$L\frac{d^2i}{dt^2} + \left(\frac{R}{L}\right)\frac{di}{dt} + \left(\frac{1}{LC}\right)i = 0 \quad (4)$$

Now , Applying laplace transform to this equation, let us assume that the solution of this equation is

$i(t) = Ke^{st}$ where K and s are constants which may be real , imaginary or complex.

Now, from eq^n (4),

$$LKs^2e^{st} + RKe^{st} + \frac{1}{C}Ke^{st} = 0$$

Which on simplification gives,

Or $s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC} = 0$

The roots of the equation would be

$$s_1, s_2 = \frac{R}{2L} \pm \sqrt{\left(\frac{R^2}{4L^2}\right) - \left(\frac{1}{LC}\right)}$$

The general solution of the differential equation is thus,

$$i(t) = K_1e^{s_1t} + K_2e^{s_2t}$$

where K1 and K2 are determined from the initial conditions.

Now , if we define,

$\alpha = Damping\ Coefficient = \frac{R}{2L}$

and natural frequency , $\omega_n = \frac{1}{\sqrt{LC}}$ which is also

known as undamped natural frequency or resonant frequency.

Thus roots are : $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - {\omega_n}^2}$

The final form of solution depends on whether

$\left(\frac{R^2}{4L^2}\right) > \frac{1}{LC}$; $\left(\frac{R^2}{4L^2}\right) = \frac{1}{LC}$ and $\left(\frac{R^2}{4L^2}\right) < \frac{1}{LC}$

The three cases can be analysed based on the initial

conditions of the circuit which are : overdamped case if

$\alpha > \omega_n$ Critically damped case if $\alpha = \omega_n$ and under damped case if $\alpha < \omega_n$. [2]

3.2 Application in control system

The control system can be classified as electrical, mechanical, hydraulic, thermal and so on. All system can be described by integro-differential equations of various orders. While the output of such systems for any input can be obtained by solving such integro-differential equations. Mathematically, it is very difficult to solve such equations in time domain. The laplace transform of such integro-differential equations converts them into simple algebraic equations. All the complicated computations then can be easily transformed equations are known as equations in frequency domain.

Then by eliminating unwanted variable, the required variable in s domain can be obtained. Then by using technique of laplace inverse , time domain function for the required variable can be obtained. Hence making the computations easy by converting the integro-differential equations into algebraic is the main essence of the laplace transform.[4]

Example 1 Find the transfer function of the given circuit.[4]

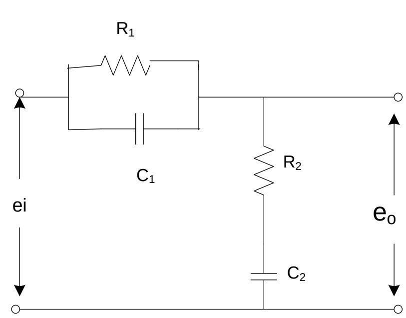

Fig.: 2

Solution,

Draw the Laplace domain network of the given system

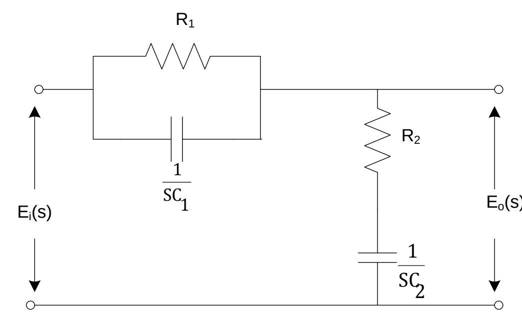

Fig. : 3

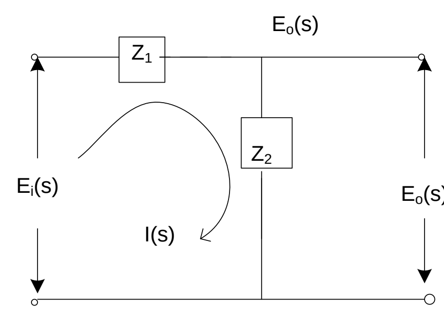

Fig. : 4

$$Z_1 = R_1 \parallel \frac{1}{sC_1} = \frac{R_1 \times \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} = \frac{R_1}{1 + sR_1C_1}$$

$$Z_2 = R_2 + \frac{1}{sC_2} = \frac{1 + sR_2C_2}{sC_2}$$

$$I(s) = \frac{E_i(s) - E_0(s)}{Z_1} \quad \text{-------(1)}$$

$$E_0(s) = I(s)Z_2 \quad \text{------(2)}$$

$$\therefore E_0(s) = \frac{E_i(s) - E_0(s)}{Z_1} \times Z_2$$

$$\therefore E_0(s)\left[1 + \frac{Z_2}{Z_1}\right] = \frac{Z_2}{Z_1} E_i(s)$$

$$\therefore \frac{E_0(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{\frac{1 + sR_2C_2}{sC_2}}{\frac{R_1}{1 + sR_1C_1} + \frac{(1 + sR_2C_2)}{sC_2}}$$

$$\therefore \frac{E_0(s)}{E_i(s)} = \frac{(1 + sR_2C_2)(1 + sR_1C_1)}{s^2R_1R_2C_1C_2 + s[R_1C_1 + R_2C_2 + R_1C_2] + 1}$$

4. PROPERTIES OF FOURIER TRANSFORM

The properties of Fourier transform are as follows [3]

4.1 Linearity

If $F_1(s)$ and $F_2(s)$ are Fourier transform of $f_1(x)$ *and* $f_2(x)$ respectively then

$$F[af_1(x)+bf_2(x)]=aF_1(s)+bF_2(s)$$

4.2 Change of scale property

If $F(s)$ is the complex Fourier Transform of f(x) then

$$F\{f(ax)\}=\frac{1}{a}F\left(\frac{s}{a}\right)$$

4.3 Shifting property

If $F(s)$ is the complex Fourier Transform of f(x) then

$$F\{f(x-a)\}=e^{isa}F(s)$$

5. APPLICATION OF FOURIER TRANSFORM

Fourier transform can be used in communications, linear system analysis, statistics, quantum physics,optics, solution of partial differential equations and antennas, etc.[5]

6. CONCLUSION

In this paper the applications of laplace transform in areas of electrical power engineering and control system are given. Also the Laplace transform resolves a function into its moments whereas the Fourier transform resolves functions or signal into its mode of vibration.

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